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On uniqueness of large solutions of nonlinear parabolic equations in nonsmooth domains

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Abstract We study the existence and uniqueness of the positive solutions of the problem (P): $\partial_t u - \Delta u + u^q = 0$ $(q > 1)$ in $\Omega \times (0, \infty)$, $u = \infty$ on $\partial\Omega \times (0, \infty)$ and $u(., 0) \in L^1(\Omega)$, when Ω is a bounded domain in $\mathbb{R}^N$. We construct a maximal solution, prove that this maximal solution is a large solution whenever $q < N/(N-2)$ and it is unique if $\partial\Omega = \partial\overline{\Omega}^c$. If $\partial\Omega$ has the local graph property, we prove that there exists at most one solution to problem (P).

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1 Introduction

Let $q > 1$ and let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega := \Gamma$. It has been proved by Keller [5] and Osserman [11] that there exists a *maximal solution* $\overline{u}$ to the stationnary equation

$$-\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega. \tag{1.1}$$

When $1 < q < N/(N-2)$ this maximal solution is a *large solution* in the sense that

$$\lim_{\rho(x)\to 0} \overline{u}(x) = \infty \tag{1.2}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. Furthermore Véron proves in [12] that $\overline{u}$ is the unique large solution whenever $\partial\Omega = \partial\overline{\Omega}^c$. When $q \geq N/(N-2)$ his proof of uniqueness does not apply. Marcus and Véron prove in [7] that, there exists at most one large solution, provided $\partial\Omega$ is locally the graph of a continuous function. The aim of this article is to extend these questions to the parabolic equation

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega \times (0, \infty). \tag{1.3}$$

We are interested into positive solutions which satisfy

$$\lim_{t\to 0} u(., t) = f \quad \text{in } L^1_{loc}(\Omega), \tag{1.4}$$

where $f \in L^1_{loc+}(\Omega)$ and

$$\lim_{(x,t)\to(y,s)} u(x,t) = \infty \quad \forall (y,s) \in \Gamma \times (0,\infty). \tag{1.5}$$

Notice that if the initial and boundary conditions are exchanged, i.e. $u(.,t)$ blows-up when $t \to 0$ and coincides with a locally integrable function on $\Gamma \times (0,\infty)$, this problem is associated with the study of the initial trace, and much work has been done by Marcus and Véron [9] in the case of a smooth domain. In particular they obtain the existence and uniqueness when q is subcritical, i.e. $1 < q < 1 + 2/N$.

In this article we prove two series of results:

Theorem A *Assume $q > 1$ and Ω is a bounded domain. Then for any $f \in L^1_{loc+}(\Omega)$ there exists a maximal solution $\overline{u}_f$ to problem (2.5) satisfying (1.4). If $1 < q < N/(N-2)$, $\overline{u}_f$ satisfies (1.5). At end, if $1 < q < N/(N-2)$ and $\partial\Omega = \partial\overline{\Omega}^c$, $\overline{u}_f$ is the unique solution of the problem which satisfies (1.5).*

The proof of uniqueness is based upon the construction of self-similar solutions of (2.5) in $\mathbb{R}^N \setminus \{0\} \times (0,\infty)$, with a persistent strong singularity on the axis $\{0\} \times (0,\infty)$ and a zero initial trace on $\mathbb{R}^N \setminus \{0\}$. This solution, which is studied in Appendix, is reminiscent of the very singular solution of Brezis, Peletier and Terman [2], although the method of construction is far different. The uniqueness is a delicate adaptation to the parabolic framework of the proof by contradiction of [12].

Theorem B *Assume $q > 1$, Ω is a bounded domain and $\partial\Omega$, is locally a continuous graph. Then for any $f \in L^1_{loc+}(\Omega)$ there exists at most one solution to problem (2.5) satisfying (1.4) and (1.5).*

For proving this result, we adapt the idea which was introduced in [7] of constructing local super and subsolutions by small translations of the domain, but the non-uniformity of the boundary blow-up creates an extra-difficulty. In an appendix we study a self-similar equation which plays a key-role in our construction,

$$\left\{ \begin{array}{r} H'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right) H' + \dfrac{1}{q-1} H - |H|^{q-1} = 0 \\ \lim_{r\to 0} H(r) = \infty \\ \lim_{r\to\infty} r^{2/(q-1)} H(r) = 0. \end{array} \right. \tag{1.6}$$

We prove the existence and the uniqueness of the positive solution of (1.6) when $1 < q < N/(N-2)$ and we give precise asymptotics when $r \to 0$ and $r \to \infty$.

This article is organised as follows: 1- Introduction. 2- The maximal solution 3- The case $1 < q < N/(N-2)$. 4- The local continuous graph property. 5- Appendix.

2 The maximal solution

In this section Ω is an open domain of $\mathbb{R}^N$, with a compact boundary $\Gamma := \partial\Omega$. If G is any open subset of $\mathbb{R}^N$ and $0 < T \le \infty$, we denote $Q^G_T := G \times (0,T)$. If $f \in L^1_{loc+}(\Omega)$, we

consider the problem

$$\left\{\begin{array}{l}\partial_t u-\Delta u+|u|^{q-1}u=0 \quad \text{in } Q_\infty^\Omega\\ \qquad\quad \lim_{t\to 0}u(.,t)=f(.) \quad \text{in } L^1_{loc}(\Omega)\\ \lim_{(x,t)\to(y,s)}u(x,t)=\infty \quad \forall (y,s)\in\Gamma\times(0,\infty).\end{array}\right. \tag{2.1}$$

By the next result, we reduce the lateral blow-up condition by a locally uniform one in which we set $\rho(x)=\operatorname{dist}(x,\Gamma)$.

Lemma 2.1 *The following two conditions are equivalent*

$$\lim_{(x,t)\to(y,s)}u(x,t)=\infty \quad \forall (y,s)\in\Gamma\times(0,\infty) \tag{2.2}$$

and

$$\lim_{\rho(x)\to 0}u(x,t)=\infty \quad \textit{uniformly on } [\tau,T], \tag{2.3}$$

for any $0<\tau<T<\infty$.

Proof. It is clear that (2.3) is equivalent to the fact that (2.2) holds uniformly on $\Gamma\times[\tau,T]$. By contradiction, we assume that (2.2) does not hold uniformly for some $T>\tau>0$. Then there exists $\beta>0$ such that for any $\delta>0$, there exist two couples $(y_\delta,s_\delta)\in\Gamma\times[\tau,T]$ and $(x_\delta,t_\delta)\in\Omega\times[\tau,T]$ such that

$$|x_\delta-y_\delta|+|t_\delta-s_\delta|\leq\delta \quad \text{and} \quad u(x_\delta,t_\delta)\leq\beta. \tag{2.4}$$

Taking $\delta=1/n$, $n\in\mathbb{N}^*$, we can assume that $\{\delta\}$ is discrete and that $y_\delta\to y\in\Gamma$ and $s_\delta\to s\in[\tau,T]$. Thus $x_\delta\to y$ and $t_\delta\to s$. Therefore (2.4) contradicts (2.2). $\square$

Theorem 2.2 *For any $q>1$ and $f\in L^1_{loc\,+}(\Omega)$, there exists a maximal solution $u:=\overline{u}_f$ of*

$$\partial_t u-\Delta u+|u|^{q-1}u=0 \quad \textit{in } Q_\infty^\Omega \tag{2.5}$$

which satisfies

$$\lim_{t\to 0}u(.,t)=f(.) \quad \textit{in } L^1_{loc}(\Omega). \tag{2.6}$$

Proof. Let Ω_n be an increasing sequence of smooth bounded domains such that $\overline{\Omega}_n\subset\Omega_{n+1}\subset\Omega$ and $\cup\Omega_n=\Omega$. For each n let $u_{n,f}$ be the increasing limit when $k\to\infty$ of the $u_{n,k,f}$ solution of

$$\left\{\begin{array}{l}\partial_t u_{n,k,f}-\Delta u_{n,k,f}+u_{n,k,f}^q=0 \quad \text{in } Q_\infty^{\Omega_n}\\ \qquad\quad u_{n,k,f}(x,t)=k \quad \text{in } \partial\Omega_n\times(0,\infty)\\ \qquad\quad u_{n,k,f}(x,0)=f\chi_{\Omega_n} \quad \text{in } \Omega_n.\end{array}\right. \tag{2.7}$$

By the maximum principle and a standard approximation argument $n\mapsto u_{n,k,f}$ is decreasing thus $n\mapsto u_{n,f}$ too. The limit $\overline{u}_f$ of the $u_{n,f}$ satisfies (2.5) and (2.6). It is independent of the exhaustion $\{\Omega_n\}$ of Ω. Let u be a positive solution of (2.5) in Q_∞^Ω which satisfies (2.6). Since the initial trace of u is a locally integrable function, $u^q\in L^1_{loc}(\Omega\times[0,\infty))$. By

Fubini we can assume that, for any n, $u \in L^1_{loc}(\partial\Omega_n \times [0,\infty))$. Because $(u - u_{n,k,f})_+ \leq u$ and tends to 0 when $k \to \infty$, it follows by Lebesgue's theorem that

$$\lim_{k\to\infty} \|(u - u_{n,k,f})_+\|_{L^1(\partial\Omega_n\times(0,T))} = 0 \quad \forall T > 0.$$

Applying the maximum principle in $\Omega_n \times (0,\infty)$ yields to

$$u \leq \lim_{k\to\infty} u_{n,k,f} = u_{n,f} \Longrightarrow u \leq \lim_{n\to\infty} u_{n,f} = \overline{u}_f.$$

□

Theorem 2.3 *For any $q > 1$ and $f \in L^1_{loc+}(\Omega)$, there exists a minimal nonnegative solution $\underline{u}_f$ of (2.5) in Q^Ω_∞ which satisfies (2.6).*

Proof. The scheme of the construction is similar to the one of $\overline{u}_f$: with the same exhaustion $\{\Omega_n\}$ of Ω, we consider the solution $u_{n,0,f}$ solution of

$$\left\{\begin{array}{l} \partial_t u_{n,0,f} - \Delta u_{n,0,f} + u^q_{n,0,f} = 0 \quad \text{in } Q^{\Omega_n}_\infty \\ \qquad u_{n,0,f}(x,t) = 0 \quad \text{in } \partial\Omega_n \times (0,\infty) \\ \qquad u_{n,0,f}(x,0) = f\chi_{\Omega_n} \quad \text{in } \Omega_n. \end{array}\right. \tag{2.8}$$

By the maximum principle, $n \mapsto u_{n,0,f}$ is increasing and dominated by $\overline{u}_f$. Therefore it converges to some solution $\underline{u}_f$ of (2.5), which satisfies (2.6) as $u_{n,0,f}$ and $\overline{u}_f$ do it. Using the same argument as in the proof of Theorem 2.2, there holds $u_{n,0,f} \leq u$ in $Q^{\Omega_n}_\infty$ for a suitable exhaustion. Thus $\underline{u}_f \leq u$. □

Remark. Because of the lack of regularity of $\partial\Omega$, there is no reason for $\overline{u}_f$ (resp $\underline{u}_f$) to tend to infinity (resp. zero) on $\partial\Omega \times (0,\infty)$.

The next statement will be very usefull for proving uniqueness results.

Theorem 2.4 *Assume $q > 1$, $f \in L^1_{loc+}(\Omega)$ and u_f is a nonnegative solution of (2.5) satisfying (2.6). Then there exists a nonnegative solution u_0 of (2.5) satisfying*

$$\lim_{t\to 0} u_0(.,t) = 0 \quad \textit{in } L^1_{loc}(\Omega), \tag{2.9}$$

such that

$$0 \leq u_f - \underline{u}_f \leq u_0 \leq u_f, \tag{2.10}$$

and

$$0 \leq \overline{u}_f - u_f \leq \overline{u}_0 - u_0. \tag{2.11}$$

Proof. Step 1: construction of u_0. The function $w = u_f - \underline{u}_f$ is a nonnegative subsolution of (2.5) which satisfies

$$\lim_{t\to 0} w(.,t) = 0 \quad \text{in } L^1_{loc}(\Omega).$$

Using the above considered exhaustion of Ω, we denote by v_n the solution of

$$\left\{\begin{array}{l} \partial_t v_n - \Delta v_n + v_n^q = 0 \quad \text{in } Q^{\Omega_n}_\infty \\ \qquad v_n(x,t) = u_f - \underline{u}_f \quad \text{in } \partial\Omega_n \times (0,\infty) \\ \qquad v_n(x,0) = 0 \quad \text{in } \Omega_n. \end{array}\right. \tag{2.12}$$

By the maximum principle

$$u_f - \underline{u}_f \leq v_n \leq u_f \quad \text{in } Q_\infty^{\Omega_n}.$$

Therefore $v_{n+1} \geq v_n$ on $\partial\Omega_n \times (0,\infty)$; this implies that the same inequality holds in $Q_\infty^{\Omega_n}$. If we denote by u_0 the limit of the $\{v_n\}$, it is a solution of (2.5) in Q_∞^{Ω}. For any compact $K \in \Omega$, there exists n_K and $\alpha > 0$ such that dist $(K, \Omega_n^c) \geq \alpha$ for $n \geq n_K$ therefore v_n remains uniformly bounded on K by Brezis-Friedman estimate [3]. Thus the local equicontinuity of the v_n (consequence of the regularity theory for parabolic equations) implies that u_0 satisfies (2.9).

Step 2: proof of (2.11). We follow a method introduced in [8] in a different context. For $n \in \mathbb{N}$ and $k > 0$ fixed, we set

$$Z_{f,n} = u_{f,n} - u_f \quad \text{and} \quad Z_{0,n} = u_{0,n} - u_0,$$

where we assume that the n are chosen such that $u_f, u_0 \in L^1_{loc}(\partial\Omega_n \times [0,\infty))$, and

$$\phi(r,s) = \begin{cases} \dfrac{r^q - s^q}{r-s} & \text{if } r \neq s \\ 0 & \text{if } r = s. \end{cases}$$

By convexity,

$$\begin{cases} r_0 \geq s_0, \; r_1 \geq s_1 \\ r_1 \geq r_0, \; s_1 \geq s_0 \end{cases} \Longrightarrow \phi(r_1, s_1) \geq \phi(r_0, s_0).$$

Therefore

$$\phi(u_{f,n}, u_f) \geq \phi(u_{0,n}, u_0) \quad \text{in } Q_T^{\Omega_n},$$

and

$$\begin{aligned} 0 &= \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + u_{f,n}^q - u_f^q - u_{0,n}^q + u_0^q \\ &= \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f) Z_{f,n} - \phi(u_{0,n}, u_0) Z_{0,n}, \end{aligned}$$

which implies

$$\partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)(Z_{f,n} - Z_{0,n}) \leq 0.$$

But $Z_{f,n} - Z_{0,n} = 0$ in $\Omega_n \times \{0\}$ and

$$\int_0^\infty \int_{\partial\Omega_n} |Z_{f,n} - Z_{0,n}| \, dS \, dt = 0$$

by approximations. By the maximum principle $Z_{f,n,k} - Z_{0,n,k} \leq 0$. Letting $n \to \infty$ yields to

$$\overline{u}_f - u_f \leq \overline{u}_0 - u_0,$$

which ends the proof. □

3 The case $1 < q < N/(N-2)$

In this section we assume that Ω is a domain of $\mathbb{R}^N$ with a compact boundary. We first prove that the maximal solution is a large solution

Theorem 3.1 *Assume $1 < q < N/(N-2)$ and $f \in L^1_{loc+}(\Omega)$. Then the maximal solution $\overline{u}_f$ of (2.5) in Q_T^Ω which satisfies (2.6) satisfies also (2.3).*

Proof. In Appendix we construct the self-similar solution $V := V_N$ of (2.5) in $Q_\infty^{\mathbb{R}^N\setminus\{0\}}$ which has initial trace zero in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$\lim_{|x|\to 0} V_N(x,t) = \infty,$$

locally uniformly on $[\tau,\infty)$, for any $\tau > 0$. Furthermore $V_N(x,t) = t^{-1/(q-1)} H_N(|x|/\sqrt{t})$. If $a \in \partial\Omega$, the restriction to Ω_n of the function $V_N(x-a,t)$ is bounded from above by $u_{n,f}$. Letting $n \to \infty$ yields to

$$V_N(x-a,t) \le \overline{u}_f(x,t) \quad \forall (x,t) \in Q_\infty^\Omega. \tag{3.1}$$

If we consider $x \in \Omega$ and denote by a_x a projection of x onto $\partial\Omega$, there holds

$$t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) = V_N(x-a_x,t) \le \overline{u}_f(x,t). \tag{3.2}$$

Using (5.2), we derive that $\overline{u}_f$ satisfies (2.3). □

Theorem 3.2 *Assume $1 < q < N/(N-2)$, $f \in L^1_{loc+}(\Omega)$ and $\partial\Omega = \partial\overline{\Omega}^c$. Then $\overline{u}_f$ is the unique solution of (2.5) in Q_T^Ω which satisfies (2.6) and (2.3).*

Proof. Assume that u_f is a solution of (2.5) in Q_T^Ω such that (2.6) and (2.3) hold. By Theorem 2.4 there exists a positive solution u_0 with zero initial trace such that

$$0 \le u_f - u_0 \le \underline{u}_f \tag{3.3}$$

and (2.11) are satisfied. Since $\underline{u}_f(x,t) \le ((q-1)t)^{-1/(q-1)}$ (notice that this last expression is the maximal solution of (2.5) in $Q_\infty^{\mathbb{R}^N}$), the function u_0 satisfies also (2.3). Therefore, it is sufficient to prove that $\overline{u}_0 = u_0 := u$.

Step 1: bilateral estimates. Since $\partial\Omega = \partial\overline{\Omega}^c$, for any $a \in \partial\Omega$, there exists a sequence $\{a_n\} \subset \overline{\Omega}^c$ converging to a. If u is any solution of (2.5) in Q_T^Ω which satisfies (2.3) and (2.9), there holds

$$V_N(x-a_n,t) \le u(x,t) \Longrightarrow V_N(x-a,t) \le u(x,t).$$

In particular, if $a = a_x$, we see that u satisfies (3.2). In order to obtain an estimate from above we consider for $r < \rho(x)$ the solution $(y,t) \mapsto u_{x,r}(y,t)$ of

$$\begin{cases} \partial_t u_{x,r} - \Delta u_{x,r} + u_{x,r}^q = 0 & \text{in } Q_\infty^{B_r(x)} \\ \lim_{(y,t)\to(z,0)} u_{x,r}(y,t) = 0 & \forall z \in B_r(x) \\ \lim_{|x|\uparrow r} u_{x,r}(x,t) = \infty & \text{locally uniformly on } [\tau,\infty), \text{ for any } \tau > 0 \end{cases} \tag{3.4}$$

Then

$$\overline{u}_0(y,t) \leq u_{x,r}(y,t) \Longrightarrow \overline{u}_0(y,t) \leq u_{x,\rho(x)}(y,t) \quad \forall (y,t) \in Q_\infty^{B_{\rho(x)}(x)}.$$

In particular, with $u_{0,r} = u_r$,

$$\overline{u}_0(x,t) \leq u_{\rho(x)}(0,t) = (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2).$$

Therefore

$$t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) \leq u(x,t) \leq \overline{u}_0(x,t) \leq (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2). \tag{3.5}$$

The function $s \mapsto u_1(0,s)$ is increasing by the same argument as the one of Corollary 4.3 and bounded from above by the unique solution P of

$$\left\{ \begin{array}{l} -\Delta P + P^q = 0 \quad \text{in } B_1 \\ \lim_{|x|\to 1} P(x) = \infty. \end{array} \right. \tag{3.6}$$

Therefore it converges to P locally uniformly in B_1 and $\lim\limits_{s\to\infty} u_1(0,s) = P(0)$. Thus

$$t/(\rho(x))^2 \to \infty \Longrightarrow (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2) \approx P(0)(\rho(x))^{-2/(q-1)}. \tag{3.7}$$

On the other hand, if $t/(\rho(x))^2 \to \infty$, equivalently $\rho(x)/\sqrt{t} \to 0$,

$$t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) \approx \lambda_{N,q} t^{-1/(q-1)} (\rho(x)/\sqrt{t})^{-2/(q-1)} = \lambda_{N,q} (\rho(x))^{-2/(q-1)}, \tag{3.8}$$

by (5.4).

Next, in order to obtain an estimate from above of $u_1(0,s)$ when $s \to 0$, we compare u_1 to a solution u_Θ of (2.5) in Q_∞^Θ, where Θ is a polyhedra inscribed in B_1; this polyhedra is a finite intersection of half spaces Γ_i containing Π. In each of the half space Γ_i, with boundary γ_i, we can consider the solution W_i of (2.5) in $Q_\infty^{\Gamma_i}$ which tends to infinity on $\gamma_i \times (0,\infty)$ and has value 0 on $\Gamma_i \times \{0\}$. This solution depends only on the distance to γ_i and t. Thus it is expressed by the function V_1 defined in Proposition 5.1 when $N = 1$. Moreover, since a sum of solutions is a super solution,

$$u_1 \leq u_\Theta \leq \sum_i W_i \Longrightarrow u_1(0,s) \leq \sum_i H_1(\operatorname{dist}(0,\gamma_i)/\sqrt{s}). \tag{3.9}$$

We can choose the hyperplanes γ_i such that for any $\delta \in (0,1)$, there exists $C_\delta \in \mathbb{N}_*$ such that

$$u_1(0,s) \leq C_\delta H_1((1-\delta)/\sqrt{s}). \tag{3.10}$$

Using (5.3) we derive

$$u(x,t) \geq c_{N,q} (\rho(x))^{2/(q-1)-N} t^{N/2-1/(q-1)} e^{-(\rho(x))^2/4t},$$

when $\rho(x)/\sqrt{t} \to \infty$, and

$$\overline{u}_0(x,t) \leq C H_1((1-\delta)\rho(x)/\sqrt{t}) \leq C(1-\delta)^{2/(q-1)-1} (\rho(x))^{2/(q-1)-1} t^{1/2-1/(q-1)} e^{-((1-\delta)\rho(x))^2/4t}.$$

Therefore, there exists $\theta > 1$ such that

$$\overline{u}_0(x,t) \le C(\rho(x))^{2/(q-1)-N} t^{N/2-1/(q-1)} e^{-(\rho(x))^2/4\theta t} \le Cu(x,\theta t), \tag{3.11}$$

when $\rho(x)/\sqrt{t} \to \infty$. Finally, when $m^{-1} \le \rho(x)/\sqrt{t} \le m$ for some $m > 1$, (3.5) shows that $(\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2)$ and $t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t})$ are comparable. In conclusion, there exist constants $C > P(0)/\lambda_{N,q} > 1$ and $\theta > 1$ such that

$$u(x,t) \le \overline{u}_0(x,t) \le Cu(x,\theta t) \quad \forall (x,t) \in Q^{\Omega}_{\infty}. \tag{3.12}$$

Step 2: End of the proof. Let $\tau > 0$ and $C' > C$ be fixed. The function

$$t \mapsto u_\tau(x,t) := C' u(x, t+\theta\tau)$$

is a supersolution of (2.5) in $\Omega \times (0,\infty)$ which satisfies $u_\tau(x,0) = C'u(x,\theta\tau) > \overline{u}_0(x,\tau)$ by (3.12). Furthermore,

$$C'u(x,t+\theta\tau) \ge C'(t+\theta\tau)^{-1/(q-1)} H_N(\rho(x)/\sqrt{t+\theta\tau}) = C'\lambda_{N,q}(1+o(1))(\rho(x))^{-2/(q-1)},$$

as $\rho(x) \to 0$, locally uniformly for $t \in [0,\infty)$. Similarly,

$$\overline{u}_0(x,t+\tau) \le (\rho(x))^{-2/(q-1)} u_1(0,(t+\tau)/(\rho(x))^2) = P(0)(1+o(1))(\rho(x))^{-2/(q-1)},$$

as $\rho(x) \to 0$, and also locally uniformly for $t \in [0,\infty)$. Therefore $(\overline{u}_0(x,t) - u_\tau(x,t))_+$ vanishes in a neighborhood of $\partial\Omega \times [0,T]$ for any $T > 0$. By the maximum principle

$$u_\tau(x,t) \ge \overline{u}_0(x,t) \quad \forall (x,t) \in \Omega \times (0,\infty).$$

Letting $\tau \to 0$ and $C' \to C$ yields to

$$u(x,t) \le \overline{u}_0(x,t) \le Cu(x,t) \quad \forall (x,t) \in Q^{\Omega}_{\infty}. \tag{3.13}$$

The conclusion of the proof is contradiction, following an idea introduced in [8] and developped by [12] in the elliptic case. We assume $u \ne \overline{u}_0$, thus $u < \overline{u}_0$. By convexity the function

$$w = u - \frac{1}{2C}(\overline{u}_0 - u)$$

is a supersolution and $w < u$. Moreover $w > w' := ((1+C)/2C)u$ and w' is a subsolution. Consequently, there exists a solution u_1 of (2.5) which satisfies

$$w' < u_1 \le w \Longrightarrow \overline{u}_0 - u_1 \ge \left(1 + K^{-1}\right)(\overline{u}_0 - u) \quad \text{in } Q^{\Omega}_{\infty}. \tag{3.14}$$

Notice that u_1 satisfies (2.9) and (2.3), therefore it satisfies (3.13) as u does it. Replacing u by u_1 and introducing the supersolution

$$w_1 = u_1 - \frac{1}{2C}(\overline{u}_0 - u_1)$$

and the subsolution $w_1' := ((1+C)/2C)u_1$ we see that there exists a solution u_2 of (2.5) such that

$$w_1' < u_2 \le w_1 \Longrightarrow \overline{u}_0 - u_2 \ge \left(1 + K^{-1}\right)^2 (\overline{u}_0 - u) \quad \text{in } Q^{\Omega}_{\infty}. \tag{3.15}$$

By induction, we construct a sequence of positive solutions u_k of (2.5), subject to (2.9) and (2.3) such that

$$\overline{u}_0 - u_k \ge \left(1 + K^{-1}\right)^k (\overline{u}_0 - u) \quad \text{in } Q^{\Omega}_{\infty}. \tag{3.16}$$

This is clearly a contradiction since $\left(1+K^{-1}\right)^k \to \infty$ as $k \to \infty$ and $\overline{u}_0$ is locally bounded in Q^{Ω}_{∞}. $\square$

4 The local continuous graph property

In this section, we assume that $\partial\Omega$ is compact and is locally the graph of a continuous function, which means that there exists a finite number of open sets Ω_j $(j = 1, ..., k)$ such that $\Gamma \cap \Omega_j$ is the graph of a continuous function. Our main result is the following

Theorem 4.1 *Assume $q > 1$ and $f \in L^1_{loc+}(\Omega)$. Then there exists at most one positive solution of (2.5) in Q^Ω_∞ satisfying (2.6) and (2.3).*

Suppose u_f satisfies (2.5) in Q^Ω_∞ satisfying (2.6) and (2.3), then clearly the maximal solution $\overline{u}_f$ endows the same properties. In order to prove that $u_f = \overline{u}_f$, we can assume that $f = 0$ by Theorem 2.4. We denote by u this large solution with zero initial trace. We consider some $j \in \{1, ..., k\}$, perform a rotation, denote by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ the coordinates in $\mathbb{R}^N$ and represent $\Gamma \cap \Omega_j$ as the graph of a continuous positive function ϕ defined in $C = \{x' \in \mathbb{R}^{N-1} : |x'| \leq R\}$. We identify C with $\{x = (x', 0) : |x'| \leq R\}$ and set

$$\Gamma_1 = \{x = (x', \phi(x')) : x' \in C\},$$

$$\Gamma_2 = \{x = (x', x_N) : x' \in \partial C, \, 0 \leq x_N < \phi(x'), \},$$

and

$$G_R = \{x \in \mathbb{R}^N : |x'| < R, \, 0 < x_N < \phi(x')\}.$$

We can assume that $\overline{G}_R \subset \Omega \cup \Gamma_1$,

$$\inf\{\phi(x') : x' \in C\} = R_0 > 0 \quad \text{and} \quad \sup\{\phi(x') : x' \in C\} = R_1 > R_0.$$

For $\sigma > 0$, small enough, we consider $\phi_\sigma \in C^\infty(C)$ satisfying

$$\phi(x') - \sigma/2 \leq \phi_\sigma(x') \leq \phi(x') + \sigma/2 \quad \forall x' \in C,$$

and set

$$G_{\sigma,R} = \{x \in \mathbb{R}^N : |x'| < R, \, 0 < x_N < \phi_\sigma(x') - \sigma\}$$

and

$$G'_{\sigma,R} = \{x \in \mathbb{R}^N : |x'| < R, \, 0 < x_N < \phi_\sigma(x') + \sigma\}.$$

The upper boundaries of G_σ and G'_σ are defined by

$$\Gamma_{1,\sigma} = \{x = (x', \phi_\sigma(x') - \sigma) : x' \in C\},$$

$$\Gamma'_{1,\sigma} = \{x = (x', \phi_\sigma(x') + \sigma) : x' \in C\},$$

and the remaining boundaries are

$$\Gamma_{2,\sigma} = \{x = (x', x_N) : x' \in \partial C, \, 0 \leq x_N \leq \phi_\sigma(x') - \sigma\},$$

$$\Gamma'_{2,\sigma} = \{x = (x', x_N) : x' \in \partial C, \, 0 \leq x_N \leq \phi_\sigma(x') + \sigma\}.$$

In order to have the monotonicity of the domains, we can also assume

$$\phi_\sigma(x') - \sigma < \phi_{\sigma'}(x') - \sigma' < \phi_{\sigma'}(x') + \sigma' < \phi_\sigma(x') + \sigma \quad \forall\, 0 < \sigma' < \sigma \quad \forall\, x' \in C, \tag{4.1}$$

thus, under the condition $0 < \sigma' < \sigma$,

$$G_{\sigma,R} \subset G_{\sigma',R} \subset G_R \subset G'_{\sigma',R} \subset G'_{\sigma,R}. \tag{4.2}$$

The localization procedure is to consider the restriction of u to $Q_\infty^{G_R} := G_R \times (0,\infty)$, thus u is regular in $G_R \cup \Gamma_2 \times [0,\infty)$ and satifies

$$\lim_{x_N \to \phi(x')} u(x', x_N, t) = \infty, \tag{4.3}$$

uniformly with respect to $(x',t) \in C \times [\tau, T]$, for any $0 < \tau < T$. We construct v_σ as solution of

$$\partial_t v_\sigma - \Delta v_\sigma + v_\sigma^q = 0 \quad \text{in } Q_\infty^{G_{\sigma,R}} := G_{\sigma,R} \times (0,\infty), \tag{4.4}$$

subject to the initial condition

$$\lim_{t \to 0} v_\sigma(x,t) = 0 \quad \text{locally uniformly in } G_{\sigma,R}, \tag{4.5}$$

and the boundary conditions

$$\lim_{x_N \to \phi_\sigma(x') - \sigma} v_\sigma(x', x_N, t) = \infty \quad \forall (x',t) \in C \times (0,\infty], \tag{4.6}$$

uniformly on any set $K \times [\tau, T]$, where $T > \tau > 0$ and K is a compact subset of C and

$$v_\sigma(x,t) = 0 \quad \forall (x,t) \in \Gamma_{2,\sigma} \times [0,\infty). \tag{4.7}$$

We also construct w_σ as solution of

$$\partial_t w_\sigma - \Delta w_\sigma + w_\sigma^q = 0 \quad \text{in } Q_T^{G'_{\sigma,R}} := G'_{\sigma,R} \times (0,\infty), \tag{4.8}$$

subject to the initial condition

$$\lim_{t \to 0} w_\sigma(x,t) = 0 \quad \text{locally uniformly in } G'_{\sigma,R}, \tag{4.9}$$

and the boundary conditions

$$\left\{ \begin{array}{ll} (i) & w_\sigma(x,t) = 0 \quad \forall (x,t) \in \Gamma'_{1,\sigma} \times [0,T], \\ (i') & \lim_{(x,s) \to (y,t)} w_\sigma(x,t) = \infty \quad \forall (y,s) \in \Gamma'_{2,\sigma} \times [0,T]. \end{array} \right. \tag{4.10}$$

The functions v_σ and w_σ inherit the following properties in which the local graph property plays a fundamental role, allowing translations of the truncated domains in the x_N-direction.

Lemma 4.2 *For $\sigma > \sigma' > 0$ there holds*

$$v_{\sigma'} \le v_\sigma \quad \textit{in } Q_\infty^{G_{\sigma,R}}, \tag{4.11}$$

$$w_{\sigma'} \le w_\sigma \quad \textit{in } Q_\infty^{G'_{\sigma',R}}, \tag{4.12}$$

$$\begin{array}{ll} (i) & v_\sigma(x', x_N - 2\sigma, t) \le u(x', x_N, t) \quad \textit{in } Q_\infty^{G_R} \\ (ii) & u(x', x_N, t) \le v_\sigma(x,t) + w_\sigma(x,t) \quad \textit{in } Q_\infty^{G_{\sigma,R}}. \end{array} \tag{4.13}$$

Proof. The inequalities (4.11) and (4.12) are the direct consequence of the fact that the domains $G_{\sigma,R}$ and $G'_{\sigma',R}$ are Lipschitz and the functions v_σ and w_σ are constructed by approximations of solutions of (2.5) with bounded boundary data. For proving (4.13)-(i), we compare, for $\tau > 0$, $u(x, t-\tau)$ and $v_\sigma(x', x_N - 2\sigma, t)$ in $Q_\infty^{G_R}$. Because u satisfies (2.3), and $v_\sigma(x', x_N - 2\sigma, 0) = 0$ in G_R, (4.13)-(i) follows by the maximum principle. The proof of (4.13)-(ii) needs no translation, but the fact that the sum of two solutions is a supersolution. □

Corollary 4.3 *There exist* $v_0 = \lim_{\sigma\to 0} v_\sigma$ *and* $w_0 = \lim_{\sigma\to 0} w_\sigma$ *and there holds*

$$v_0 \le u \le v_0 + w_0 \quad \text{in } Q_\infty^{G_R}. \tag{4.14}$$

Moreover, the functions $t \mapsto v_0(x,t)$ *and* $t \mapsto w_0(x,t)$ *are increasing on* $(0,\infty)$, $\forall x \in G_R$.

Proof. The first assertion follows from (4.11)-(4.12), and (4.14) from (4.13). Since v_0 is the limit, when $\sigma \to 0$ of v_σ which satisfy equation (4.4) in $Q_T^{G_{\sigma,R}}$, initial condition (4.5) and boundary conditions (4.6), (4.7), it is sufficient to prove the monotonicity of $t \mapsto v_\sigma(.,t)$. Moreover v_σ is the limit, when k tends to infinity of the $v_{k,\sigma}$ solutions of (2.5) in $Q_T^{G_{\sigma,R}}$, which satisfy the same boundary conditions as v_σ on $\Gamma_{2,\sigma} \times [0,T]$, the same zero initial condition and

$$\lim_{x_N \to \phi(x')-\sigma} v_{k,\sigma}(x', x_N, t) = k.$$

For $\tau > 0$, we define V_τ by $V_\tau(x,t) = (v_{k,\sigma}(x,t) - v_{k,\sigma}(x, t+\tau))_+$. Because $\partial G_{\sigma,R}$ is Lipschitz and V_τ is a subsolution of (2.5) which vanishes on $\partial G_{\sigma,R} \times [0,T]$ and at $t = 0$, it is identically zero. This implies $v_{k,\sigma}(x,t) \le v_{k,\sigma}(x, t+\tau)$, and the monotonicity property of v_0, by strict maximum principle and letting $\sigma \to 0$. The proof of the monotonicity of w_0 is similar. □

The key step of the proof is the following result.

Proposition 4.4 *Let* $\epsilon, \tau > 0$. *Then there exists* $\delta_\epsilon > 0$ *such that, if we denote*

$$G_{\delta,R'} = \{x = (x', x_N) : |x'| < R' \text{ and } \phi(x') - \delta \le x_N < \phi(x')\},$$

there holds, for $R' < R/\sqrt{N-1}$,

$$w_0(x,t) \le \epsilon v_0(x, t+\tau) \quad \forall (x,t) \in Q_\infty^{G_{\delta,R'}}. \tag{4.15}$$

Proof. Using the result in Appendix, we recall that $V := V_1$ is the unique positive and self-similar solution of the problem

$$\begin{cases} \partial_t V - \partial_{zz} V + V^q = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\ \lim_{t\to 0} V(z,t) = 0 \quad \forall z > 0 \\ \lim_{z\to 0} V(z,t) = \infty \quad \forall t > 0, \end{cases} \tag{4.16}$$

and it is expressed by $V_1(z,t) = t^{-1/(q-1)} H_1(x/\sqrt{t})$, where H_1 satisfies (5.2)-(5.3) with $N = 1$. We set $R_N = R/\sqrt{N-1}$ so that

$$C_\infty := \{x' = (x_1, ..., x_{N-1}) : \sup_{j \le N-1} |x_j| < R_N\} \subset C = \{x' : |x'| \le R\}$$

and we define

$$\tilde{w}(x,t) = W(x_N,t) + \sum_{j=1}^{N-1}(W(x_j - R,t) + W(R - x_j,t)).$$

The function $\tilde{w}$ a super solution in $\Theta \times \mathbb{R}^+$ where $\Theta := \{(x', x_N) : x' \in C_\infty, x_N > 0\}$ which blows up on

$$\{x : x_N = 0\,,\ \sup_{j \leq N-1} |x_j| \leq R\} \bigcup_{j \leq N-1} \{x : x_N \geq 0\,,\ x_j = \pm R\}\,.$$

Therefore $w_0 \leq \tilde{w}$ in $Q_T^{G_{R_N}}$. Moreover $\tilde{w}(x,t) \to 0$ when $t \to 0$, uniformly on

$$G^*_{\alpha,R'} := \{x = (x_1,x_2) : |x_1| \leq R', \alpha \leq x_2 \leq \phi(x_1)\},$$

for any $\alpha \in (0, R_0]$ and $R' \in (0, R_N)$. Since for any $\tau > 0$, $v_0(x, t+\tau) \to \infty$ when $\rho(x) \to 0$, locally uniformly on $[0,\infty)$, and $\tilde{w}(x,t)$ remains uniformly bounded on $Q_\infty^{G_{\delta,R'}}$, for any $\delta > R_0$, it follows that for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$w_0(x,t) \leq \tilde{w}(x,t) \leq \epsilon v_0(x,t+\tau) \quad \forall (x,t) \in Q_\infty^{G_{\delta_\epsilon,R'}}.$$

□

Proof of Theorem 4.1. Assume u is a solution of (2.5) satisfying (2.6) and (2.3). Then there holds in $Q_\infty^{G_{\delta_\epsilon,R'}}$,

$$v_0(.,t) \leq u(.,t) \leq v_0(.,t) + \epsilon v_0(.,t+\tau). \tag{4.17}$$

Therefore

$$v_0(.,t+\tau) \leq u(.,t+\tau) \leq v_0(.,t+\tau) + \epsilon v_0(.,t+2\tau),$$

from which follows

$$(1+\epsilon)u(.,t+\tau) \geq (1+\epsilon)v_0(.,t+\tau) \geq v_0(.,t) + \epsilon v_0(.,t+\tau)$$

since $t \mapsto v_0(.,t)$ is increasing by Corollary 4.3. The maximal solution $\overline{u}_0$ satisfies (4.17) too; consequently the following inequality is verified in $Q_\infty^{G_{\delta_\epsilon,R'}}$,

$$(1+\epsilon)u(.,t+\tau) \geq \overline{u}_0(.,t). \tag{4.18}$$

Since $\partial\Omega$ is compact, there exists $\delta^* > 0$ such that (4.18) holds whenever $t \in [0,T]$ ($T > 0$ arbitrary) and $\rho(x) \leq \delta^*$. Furthermore

$$\lim_{t \to 0} \max\{(\overline{u}_0(x,t) - (1+\epsilon)u(x,t+\tau))_+ : \rho(x) \geq \delta^*\} = 0$$

because of (2.6). Since $(\overline{u}_0(x,t) - (1+\epsilon)u(x,t+\tau))_+$ is a subsolution, which vanishes at $t = 0$ and near $\partial\Omega \times [0,T]$, it follows that (4.18) holds in Q_T^Ω. Letting $\epsilon \to 0$ and $\tau \to 0$ yields to $u \geq \overline{u}_0$. □

Remark. The existence of large solutions when $q \geq N/(N-2)$ is a difficult problem as it is already in the elliptic case. We conjecture that the necessary and sufficient conditions, obtained by Dhersin-Le Gall when $q = 2$ [4] and Labutin [6] in the general case $q > 1$, and expressed by mean of a Wiener type criterion involving the $C^{\mathbb{R}^N}_{2,q'}$-Bessel capacity, are still valid. As in [7], it is clear that if $\partial\Omega$ satisfies the exterior segment property and $1 < q < (N-1)/(N-3)$, then $\overline{u}_0$ is a large solution.

5 Appendix

The proof of this result is based upon the existence of solution of (2.5) in $Q_\infty^{\mathbb{R}^N\setminus\{0\}}$ with a persistent singularity on $\{0\}\times[0,\infty)$.

Proposition 5.1 *For any $q>1$, there exists a unique positive function $V:=V_N$ defined in $\mathbb{R}_+\times\mathbb{R}_+$ satisfying, for any $\tau>0$*

$$\left\{\begin{array}{l}\partial_t V-\Delta V+V^q=0 \quad \text{in } \; Q_\infty^{\mathbb{R}^N\setminus\{0\}}\\ \lim_{(x,t)\to(y,0)}V(x,t)=0 \quad \forall y\in\mathbb{R}^N\setminus\{0\}\\ \lim_{|x|\to 0}V(x,t)=\infty \quad \text{locally uniformly on } [\tau,\infty),\text{ for any } \tau>0\end{array}\right. \tag{5.1}$$

Then $V_N(x,t)=t^{-1/(q-1)}H_N(|x|/\sqrt{t})$, where $H:=H_N$ is the unique positive function satisfying

$$\left\{\begin{array}{l}H''+\left(\dfrac{N-1}{r}+\dfrac{r}{2}\right)H'+\dfrac{1}{q-1}H-H^q=0 \quad \text{in } \; \mathbb{R}_+\\ \lim_{r\to 0}H(r)=\infty\\ \lim_{r\to\infty}r^{2/(q-1)}H(r)=0.\end{array}\right. \tag{5.2}$$

Furthermore there holds

$$H_N(r)=c_{N,q}r^{2/(q-1)-N}e^{-r^2/4}(1+O(r^{-2})) \quad \text{as } \; r\to\infty, \tag{5.3}$$

and

$$H_N(r)=\lambda_{N,q}r^{-2/(q-1)}(1+O(r)) \quad \text{as } \; r\to 0, \tag{5.4}$$

Proof. If we assume $1<q<N/(N-2)$, the $C_{2,1,q'}$ parabolic capacity of the axis $\{0\}\times\mathbb{R}\subset\mathbb{R}^{N+1}$ is positive, therefore there exists a unique solution $u:=u_\mu$ to the problem

$$\partial_t u-\Delta u+|u|^{q-1}u=\mu \quad \in\mathbb{R}^N\times\mathbb{R}, \tag{5.5}$$

(see [1]) where μ is the uniform measure on $\{0\}\times\mathbb{R}_+$ defined by

$$\int\zeta d\mu=\int_0^\infty\zeta(0,t)dt \quad \forall\zeta\in C_0^\infty(\mathbb{R}^{N+1}).$$

If we denote $T_\ell[u](x,t)=\ell^{2/(q-1)}u(\ell x,\ell^2 t)$ for $\ell>0$, then T_ℓ leaves the equation (2.5) invariant, and $T_\ell[u_\mu]=u_{\ell^{2/(q-1)-N}\mu}$. If we replace μ by $k\mu$ ($k>0$), we obtain

$$T_\ell[u_{k\mu}]=u_{\ell^{2/(q-1)-N}k\mu}. \tag{5.6}$$

Moreover, any solution of (2.5) in $\mathbb{R}^N\setminus\{0\}\times\mathbb{R}_+$ which vanishes on $\mathbb{R}^N\setminus\{0\}\times\{0\}$ is bounded from above by the maximum solution $u:=U$ of

$$-\Delta u+u^q=0 \quad \text{in } \mathbb{R}^N\setminus\{0\}. \tag{5.7}$$

This is obtained by considering the solution U_ϵ of

$$\begin{cases} -\Delta u + u^q = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}_\epsilon \\ \lim_{|x| \to \epsilon} u(x) = \infty. \end{cases} \tag{5.8}$$

Actually,

$$U(x) := \lim_{\epsilon \to 0} U_\epsilon(x) = \lambda_{N,q} |x|^{-2/(q-1)} \quad \text{with } \lambda_{N,q} := \left[\left(\frac{2}{q-1} \right) \left(\frac{2q}{q-1} - N \right) \right]^{1/(q-1)}, \tag{5.9}$$

an expression which exists since $1 < q < N/(N-2)$. If we let $k \to \infty$ in (5.6), using the monotonicity of $\mu \mapsto u_\mu$, we obtain that $u_{k\mu} \to u_{\infty\mu}$, $u_{\infty\mu} \leq U$ and

$$T_\ell[u_{\infty\mu}] = u_{\ell^{2/(q-1)-N} \infty\mu} = u_{\infty\mu} \quad \forall \ell > 0. \tag{5.10}$$

This implies that $u_{\infty\mu}$ is self-similar, that is

$$u_{\infty\mu}(x,t) = t^{-1/(q-1)} h(x/\sqrt{t}).$$

Furthermore, $h(.)$ is positive and radial as $x \mapsto u_\mu(x,t)$ is, and it solves

$$h'' + \left(\frac{N-1}{r} + \frac{r}{2} \right) h' + \frac{1}{q-1} h - h^q = 0 \quad \text{in } \mathbb{R}_+. \tag{5.11}$$

Since $u_\mu(x,0) = 0$ for $x \neq 0$, the a priori bounds $u_{k\mu} \leq U$, the equicontinuity of the $\{u_{k\mu}\}_{k>0}$ implies that $u_{\infty\mu}(x,0) = 0$ for $x \neq 0$; therefore

$$\lim_{r \to \infty} r^{2/(q-1)} h(r) = 0. \tag{5.12}$$

The same argument as the one used in the proof of Corollary 4.3 implies that $t \mapsto u_\mu(x,t)$ is increasing, therefore $\lim_{x \to 0} u_\mu(x,t) = \infty$ for $t > 0$. This implies $\lim_{r \to 0} h(r) = \infty$. Then the proof of (5.3) follows from [10, Appendix]. When $r \to 0$, h could have two possible behaviours [13]:

(i) either

$$h(r) = \lambda_{N,q} r^{-2/(q-1)} (1 + O(r)), \tag{5.13}$$

(ii) or there exists $c \geq 0$ such that

$$h(r) = c m_N(r)(1 + O(r)), \tag{5.14}$$

where $m_N(r)$ is the Newtonian kernel if $N \geq 2$ and $m_1(r) = 1 + o(1)$.

If (ii) were true with $c > 0$ (the case $c = 0$ implying that $h = 0$ because of the behavior at ∞ and maximum principle), it would lead to

$$u_{\infty\mu}(x) = c|x|^{2-N} t^{N-2-1/(q-1)} (1 + o(1)) \quad \text{as } x \to 0, \tag{5.15}$$

for all $t > 0$. Therefore

$$\int_\epsilon^T \int_{B_1} u_{k\mu}^q dx \, dt < C(\epsilon), \tag{5.16}$$

for any $\epsilon > 0$ and $k \in (0, \infty]$. We write (5.5) under the form

$$\partial_t u_{k\mu} - \Delta u_{k\mu} = g_k + k\mu$$

where $g_k = -u_{k\mu}^q$, then $u_{k\mu} = u'_{k\mu} + u''_k$, where

$$\partial_t u'_{k\mu} - \Delta u'_{k\mu} = k\mu$$

and

$$\partial_t u''_k - \Delta u''_k = g_k.$$

By linearity $u'_{k\mu} = ku'_\mu$. Because of (5.16) u''_k remains uniformly bounded in $L^1(B_1 \times (\epsilon, T))$. This clearly contradicts $\lim_{k\to\infty} u'_{k\mu} = \infty$. Thus (5.4) holds. The proof of uniqueness is an easy adaptation of [7, Lemma 1.1]: the fact that the domain is not bounded being compensated by the strong decay estimate (5.3). This unique solution is denoted by V_N and $h = H_N$. □

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